Math 255A Lecture 26 Notes

Daniel Raban

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1 Applications of Banach-Alaoglu

1.1 Banach-Alaoglu for separable Banach spaces

Last time, we stated the following proposition.

Proposition 1.1. *If G is separable with the dense subset $\{x_1, x_2, \dots\}$, then the seminorms $\xi \mapsto |\langle x_j, \xi \rangle|$ for $\xi \in U = \{\xi : \|x\| \leq 1\}$ and $j = 1, 2, \dots$ define the same topology as $\sigma(B^*, B)$ on U.*

Proof. It sufficies to check that if $O \subseteq U$ is open for $\sigma(B^*, B)$, then O is open for the topology determined by these seminorms. Let $\xi_0 \in U$ and N be an open neighborhood of ξ_0 in $\sigma(B^*, B)$. We can assume that $N = \{\xi \in U : |\langle y_j, \xi - \xi_0 \rangle| < \varepsilon \, \forall 1 \leq j \leq M\}$. We claim that N contained a neighborhood of ξ_0 in the topology defined by the seminorms. For each $j \in \{1, \dots, M\}$, pick k_j such that $\|y_j - x_{k_j}\| < \varepsilon/4$. If $|\langle x_{k_j}, \xi - \xi_0 \rangle| < \varepsilon/2$ for $1 \leq j \leq M$, then

$$|\langle y_j, \xi - \xi_0 \rangle| \leq \underbrace{|\langle y_j - x_{k_j}, \xi - \xi_0 \rangle|}_{\leq \|y_j - x_{k_j}\| \|\xi - \xi_0\| < \varepsilon/2} + \underbrace{|\langle x_{k_j}, \xi - \xi_0 \rangle|}_{< \varepsilon/2} < \varepsilon$$

for all $\xi \in N$ and $1 \leq j \leq M$. Then $N \supseteq \{\xi \in U : |\langle x_{k_j}, \xi - \xi_0 \rangle| < \varepsilon/2, 1 \leq j \leq M\}$, and the result follows. $\square$

Corollary 1.1. *If B is separable, then $U = \{\xi \in B^* : \|\xi\| \leq 1\}$ is a complete metrizable space for $\sigma(B^*, B)$.*

Example 1.1. Let $f_n \in L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$ be bounded: $\|f_n\|_{L^p} \leq C$. Then there exists a subsequence f_{n_k} and $f' \in L^p$ such that

$$\int f_{n_k} g \, dx \to \int f g \, dx$$

for $g \in L^q$ with $1/p + 1/q = 1$.

When $p = 1$, $L^1(\mathbb{R}^d) \subseteq M(\mathbb{R}^d)$, the space of bounded measures on $\mathbb{R}^d$. If $\|f_n\|_{L^1} \leq C$, then there exists a subsequence f_{n_k} and a $\mu \in M(\mathbb{R}^d)$ such that

$$\int f_{n_k} g \, dx \to \int g d\mu$$

for all $g \in C_0(\mathbb{R}^d)$, the space of functions which vanish at ∞.

1.2 Applications of Banach-Alaoglu to minimizing functionals

Definition 1.1. A Banach space B is **reflexive** if the natural linear isometry $J : B \to B^{**}$ sending $x \mapsto (\xi \mapsto \langle x, \xi \rangle)$ is a bijection.

Proposition 1.2 (Minimization of functionals[1])**.** *Let B be a reflexive Banach space with B^* separable, and let $J : B \to \mathbb{R}$ be a function such that*

1. *J is convex: $J(\lambda u + (1 - \lambda)v) \leq \lambda J(u) + (1 - \lambda)J(v)$ fir $0 \leq \lambda \leq 1$ and $u, v \in B$.*
2. *J is norm lower semicontinuous: For all $x \in \mathbb{R}$, the set $\{u \in B : J(u) > a\}$ is open iff if $u_n \to u_0$ in B, then $J(u_0) \leq \liminf_{n\to\infty} J(u_n)$.*
3. *J is coercive: There exists $C > 1$ such that $J(u) \geq \|u\|^q/C - C$ for all u, for some $q \geq 1$.*

In particular, $\mu = \inf_{u \in B} J(u) > -\infty$. Then there exists some $u_0 \in B$ such that $J(u_0) = \mu$.

Proof. Let $u_n \in B$ be such that $J(u_n) \to \mu$. Property 3 implies that (u_n) is bounded: $\|u_n\| \leq C$. By Banach-Alaoglu, there exists a subsequence (u_{n_k}) and $u_0 \in B$ such that $u_{n_k} \to u_0$ in $\sigma(B, B^*)$. Now J is convex norm lower semicontinuous, so $\{u \in B : J(u) \leq a\}$ is closed and convex. By convexity, it is weakly closed, so J is lower semicontinuous with respect to $\sigma(B, B^*)$. If $u_{n_k} \to u_0$ in $\sigma(B, B^*)$, then $H(u_0) \leq \liminf J(u_{n_k}) = \mu$, and we get the claim. □

Here is a concrete application.

Example 1.2. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and let $H^1(\Omega) = \{u \in L^2(\Omega) : \partial_{x_j} u \in L^2(\Omega) \ \forall 1 \leq j \leq n\}$ be a Hilbert space with the inner product $\langle u, v \rangle_{H^1} = \langle u, v \rangle_{L^2} + \langle \nabla u, \nabla v \rangle_{L^2}$. Define $H_0^1(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. This is not all of H^1; roughly, $u \in H_0^1(\Omega)$ iff $u \in H^1(\Omega)$ and "$u|_{\partial\Omega} = 0$."

Apply the abstract discussion when $B = H_0^1(\Omega)$ and $J(u) = (1/2) \int_\Omega |\nabla u|^2 \, dx - \int_\Omega f u \, dx$ where $f \in L^2(\Omega)$.[2] We claim that there exists some $u_0 \in H_0$ which is a minimizer of J. Observe that:

[1] This application comes from calculus of variations.
[2] This is sometimes called the Dirichlet functional.

1. J is convex.

2. J is continuous.

3. J is coercive: $J(u) \geq \|\nabla u\|_{L^2}^2 - \|f\|_{L^2}\|u\|_{L^2} \geq \|\nabla u\|_{L^2}^2 - \varepsilon\|u\|_{L^2} - (1/\varepsilon)\|f\|_{L^2}^2$. Since $\|u\||_{L^2} \leq C\|\nabla u\|_{L^2}$ for $u \in C_0^\infty$, we get $J(u) \geq (1/C)\|u\|_H^2 - C$.

So we have a minimizer $J(u_0) \leq J(u_0 + \varepsilon v)$ for all ε. Then

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J(u_0 + \varepsilon v) = 0,$$

where $-\Delta u_0 = f$ and $u_0 \in H_0(\Omega)$.